STABILIZING A NONLINEAR CONTROLLED SYSTEM FOR STEADY-STATE MOTIONS IN THE CRITICAL CASE OF A DOUBLE ZERO ROOT

PMM Vol. 31, No. 5, 1967, pp. 963-968
N. V. STOIANOV
(Sofia, Bulgaria)
(Received January 1, 1967)

The problem of stabilizing the steady-state motions of a nonlinear controlled system in the critical case of two zero roots is considered. The problem is solved by a method based on the theory of motion stability [1 to 6] and by the techniques developed in [7 to 9]; a nonanalytic control is used. This problem is considered in [10] in the case where the double zero root is associated with one group of solutions of the first approximation of the initial system. Here we consider the second case, where two groups of solutions are associated with this root.

1. Let us consider the perturbed motion of a controlled object described by the system

$$\frac{dy}{dt} = Ay + Bu + g(y, u) \qquad (y \in \{R^{n+2}\},\ u \in \{R^m\}) \tag{1.1}$$

Here y is an $(n+2)$-dimensional perturbation vector; u is the m-dimensional vector of the control, which we assume to consist of unperturbed disturbances; A, B are constant matrices of dimensions $(n+2) \times (n+2)$ and $(n+2) \times m$; $g(y, u)$ are analytic nonlinearities in y, u. All of the coefficients of Eqs. (1.1) are assumed to be real.

Let the unperturbed motion $y = 0$ of system (1.1) for $u \equiv 0$ not be asymptotically stable. We can then pose the problem of stabilizing motion (1.1), i.e. of choosing a control $u = u(y)$ such that its substitution into (1.1) renders the unperturbed motion $y = 0$ asymptotically stable in the Liapunov sense.

Let us make use of the definition of [8] and consider the critical case of a double zero root, and specifically the second subcase in which two groups of solutions correspond to the double zero root. In the case the systems of equations of the first approximation admit of two independent linear integrals with constant coefficients. By means of the nondegenerate transformation of variables whose matrix can be constructed (see [1 to 3 and 8]) by taking these integrals as the new variables ξ, η and writing $y_i = x_i$ ($i = 1, \ldots, n$), we can reduce system (1.1) to the form

$$d\xi/dt = X(\xi, \eta, x, u), \qquad d\eta/dt = Y(\xi, \eta, x, u) \tag{1.2}$$

$$dx/dt = A_0 x + B_0 u + a\xi + b\eta + Z(\xi, \eta, x, u) \tag{1.3}$$

Here ξ, η are scalars; x is an n-vector; a, b are n-vectors; A_0 is a constant $n \times n$ matrix; B_0 is a constant $n \times m$ matrix; X, Y, Z are analytic nonlinearities in ξ, η, x, u.

After this transformation the stabilization problem for system (1.1) becomes equivalent to the same problem for system (1.2), (1.3).

As we know, the system

$$dx/dt = A_0 x + B_0 u \tag{1.4}$$

satisfies the stabilizability condition [8], so that we can construct for it a linear control of the form

$$u^0(x) = Px \tag{1.5}$$

(where P is some constant $m \times n$-matrix).

By suitable choice of control (1.5) for system (1.4), all of the eigenvalues μ_i of the matrix $A_0 + B_0P$ = const satisfy the condition

$$R_{e\mu i} < 0 \qquad (i = 1, \ldots, n) \tag{1.6}$$

Let us have a nonanalytic control of the form [9]

$$u_j(\xi, \eta, x) = u_j^\circ(x) + \sum_{p,\, q=0}^{1} \sum_{s+k=1}^{\infty} \alpha_{sk\, pq}^j \xi^s \eta^k \xi_*^p \eta_*^q \tag{1.7}$$

$$z_* = \operatorname{sign} z = \begin{cases} 1 & \text{for } z \geqslant 0 \\ -1 & \text{for } z < 0 \end{cases} \qquad \begin{pmatrix} j = 1, \ldots, m \\ s \geqslant 0,\ k \geqslant 0 \end{pmatrix}$$

Setting $p = q = 0$ in (1.7), we obtain an analytic control. In this case the stabilization problem for system (1.2), (1.3) can be solved by any of the methods of analysis known to us from stability theory [3 to 5]. Subjecting the coefficients of the series in (1.7) to the condition $\alpha_{s0}{}^j{}_{p1} = \alpha_{0k}{}^j{}_{1q} = 0$, we obtain a continuous control.

We shall consider a nonanalytic control for system (1.2), (1.3) which broadens substantially the possibilities for stabilization.

2. We shall make use of the Theorem [3] (see also [6]) known as the "reduction principle". This theorem will enable us to reduce the solution of the stability problem for complete system (1.2), (1.3) to the consideration of some "shortened" system corresponding to two zero roots. Let us formulate the reduction principle for our case. Substituting the control $u(\xi, \eta, x)$ (1.7) into system (1.2), (1.3), we obtain

$$d\xi / dt = X'(\xi, \eta, x), \qquad d\eta / dt = Y'(\xi, \eta, x) \tag{2.1}$$

$$dx / dt = (A_0 + B_0P)x + Z'(\xi, \eta, x) \tag{2.2}$$

Rejecting all terms containing x_i in the right-hand sides of Eqs. (2.1), we obtain the shortened system

$$d\xi / dt = X'(\xi, \eta, 0), \qquad d\eta / dt = Y'(\xi, \eta, 0) \tag{2.3}$$

Let us assume that in the expansions of the functions X', Y' (2.1) the lowest order of the terms dependent on x_i ($i = 1, \ldots, n$) is q, while the lowest order of these terms in x_i is $r \leq q$.

T h e o r e m 2.1. Let us assume that the unperturbed motion $\xi = \eta = 0$ for shortened system (2.3) is stable, asymptotically stable, or unstable regardless of the terms of order higher than N. Then, if the expansion of the vector $Z'(\xi, \eta, 0)$ begins with terms of order not lower than p, where

$$p \geqslant \frac{N + 1 - q + r}{r} \tag{2.4}$$

then the unperturbed motion $\xi = \eta = x_i = 0$ for complete system (2.1), (2.2) is stable, asymptotically stable, or unstable, respectively.

The proof of the theorem will not be given here, since it is essentially a repetition of the proof of the theorem for an analytic system [3].

In order to make it possible to apply the reduction principle to system (2.1), (2.2) we must reduce these equations to a form for which the expansions of the functions $Z'(\xi, \eta, 0)$ would begin with terms of sufficiently high order. To do this we carry out the Liapunov transformation

$$x_i = v_i^* + w_i \qquad (i = 1, \ldots, n) \tag{2.5}$$

Here $v^i(\xi,\eta)$ are entire rational functions of the variables ξ,η satisfying the system of partial differential equations

$$\frac{\partial v}{\partial \xi}X(\xi,\eta,v,u)+\frac{\partial v}{\partial \eta}Y(\xi,\eta,v,u)=A_0v+B_0u+a\xi+b\eta+Z(\xi,\eta,v,u) \quad (2.6)$$

where v is an n-dimensional vector. If control (1.7) is analytic (i.e. for $p=q=0$), then system (2.6) has a unique solution [3]. In the case of a nonanalytic control $u(\xi,\eta,v)$ in Formula (1.7), we substitute this control into system (2.6) and seek the solution of the latter in the form of formal series

$$v_i=\sum_{p,\,q=0}^{1}\sum_{s+k=1}^{\infty}a_{skpq}^{i}\xi^s\eta^k\xi_*{}^p\eta_*{}^q \qquad \begin{pmatrix} i=2,\ldots,n\\ s\geqslant 0,\ k\geqslant 0\end{pmatrix} \quad (2.7)$$

System (2.6) can be solved by the method of undetermined coefficients. Substituting (2.7) into (2.6) we obtain linear algebraic systems of equations for determining the coefficients a_{skpq}^{i}. The determinant $|A_0+B_0P|$ of these systems is not equal to zero by virtue of (1.6) (see [3 and 8]). These equations make it possible to determine successively all of the coefficients a_{skpq}^{i} in the expansions of the quantities v_i in such a way that Eq. (2.6) is satisfied identically in ξ,η,ξ_*,η_*. Such series (2.7) for system (2.6) always exist and are completely defined throughout the space.

If we are able to compute the continuous solutions v_i (2.7) for system (2.6) to within terms of order $s+k\leq(p-1)$, then, substituting (2.5) into (2.2) and setting $w_1\equiv 0$, we find that all of the terms of order lower than p in ξ,η must vanish. Thus, system (2.1), (2.2) can be transformed into a system which satisfies the conditions of the theorem.

On substituting Eq. (1.7) into Eqs. (2.1) and replacing the vector x by the vector v (2.7), we obtain the shortened system in the form (2.8)

$$\frac{d\xi}{dt}=X_m(\xi,\eta)+X_{m+1}(\xi,\eta)+\ldots,\qquad \frac{d\eta}{dt}=Y_m(\xi,\eta)+Y_{m+1}(\xi,\eta)+\ldots \qquad (m\geqslant 2)$$

Here $X_i(\xi,\eta)$, $Y_i(\xi,\eta)$ $(i=m,\ m+1,\ldots)$ are ith order forms given by

$$X_i(\xi,\eta)=\sum_{p,\,q=0}^{1}\sum_{s+k=i}l_{skpq}^{(i)}\xi^s\eta^k\xi_*{}^p\eta_*{}^q,\qquad Y_i(\xi,\eta)=\sum_{p,\,q=0}^{1}\sum_{s+k=i}h_{skpq}^{(i)}\xi^s\eta^k\xi_*{}^p\eta_*{}^q \quad (2.9)$$

Let us subject the coefficients of these forms to continuity conditions, i.e.

$$l_{s0p1}^{(i)}=l_{0k1q}^{(i)}=0,\quad h_{s0p1}^{(i)}=h_{0k1q}^{(i)}=0 \quad (2.10)$$

The remaining coefficients $l_{skpq}^{(i)}$, $h_{skpq}^{(i)}$ depend in a certain way on the coefficients α_{skpq}^{j} in (1.7).

System (2.8) results from system (1.2), (1.3) by way of the transformation (2.5) if we set $w_1\equiv 0$ and reject all of the equations associated with noncritical roots. Thus, according to the reducibility principle for the solution of the stability problem for complete system (1.2) and (1.3), it is sufficient to consider critical Eqs. (2.8).

Let us construct two forms of order $m+1$

$$P_{m+1}(\xi,\eta)=\xi X_m(\xi,\eta)+\eta Y_m(\xi,\eta), \quad (2.11)$$

$$Q_{m+1}(\xi,\eta)=\xi Y_m(\xi,\eta)-\eta X_m(\xi,\eta) \quad (2.12)$$

3. First let us assume that the forms $Q_{m+1}(\xi,\eta)$ are not of sign definite. In this case Eq.

$$Q_{m+1}(\xi,\eta)=\xi Y_m(\xi,\eta)-\eta X_m(\xi,\eta)=0 \quad (3.1)$$

has real solutions for ξ,η which are not simultaneously equal to zero. Eq. (3.1), whose left-hand side contains functions of the form (2.9), defines one or several broken straight lines with a salient point at the origin. Let us confine ourselves to those controls (1.7) for which transformation (2.5) is continuous and which satisfy condition (2.10) for function (2.9). Then, by virtue of the continuity of the velocity field for homogeneous equations,

$$d\xi / dt = X_m(\xi, \eta), \qquad d\eta / dt = Y_m(\xi, \eta) \tag{3.2}$$

every integral curve of these equations which passes through the origin is tangent to one of the broken straight lines (3.1). This is because, by virtue of definition (2.11) of the form $Q_{m+1}(\xi, \eta)$, on each line we have the identity

$$\xi \, d\eta / dt - \eta \, d\xi / dt = 0$$

Let us require that the condition

$$P_{m+1}(\xi, \eta) = \xi X_m(\xi, \eta) + \eta Y_m(\xi, \eta) = -\tau(\xi, \eta) \tag{3.3}$$

be fulfilled for form (2.11).

Here $\tau(\xi, \eta)$ is a positively defined function from the class (2.9) which can be taken in the form [9]

$$\tau(\xi, \eta) = \sum_{i+j=m+1} \beta_{ij}^{(m+1)} (\xi\xi_*)^i (\eta\eta_*)^j \qquad (\beta_{ij}^{(m+1)} > 0) \tag{3.4}$$

The Liapunov function satisfying the theorem on asymptotic stability can be constructed in the form

$$2V = \xi^2 + \eta^2 \tag{3.5}$$

Let us construct the total derivative of this function. By virtue of Eqs. (3.2) we obtain

$$dV / dt = P_{m+1}(\xi, \eta)$$

According to the choice of form (3.3) satisfying conditions (3.4), the derivative dV/dt assumes only negative values on each broken straight line. Thus, the following statement is valid [3 to 5].

T h e o r e m 3.1. The stabilization of system (2.8) is guaranteed by control (1.7) if the coefficients $\alpha^j_{s\lambda pq}$, β^{m+1}_{ij} can be chosen in such a way that form (2.11) is not of sign definite and that conditions (2.10), (3.4) are fulfilled.

N o t e. If $m = 2$ in (2.8), then the function $\tau(\xi, \eta)$ can be more general than (3.4), i.e.

$$\tau(\xi, \eta) = |\xi| \sum_{i+j=2} c_{ij}\xi^i\eta^j + |\eta| \sum_{i+j=2} c_{ij}^*\xi^i\eta^j \qquad (i \geqslant 0, j \geqslant 0)$$

where $|\xi| = \xi\xi_*$ is an absolute quantity, and the coefficients satisfy the Sylvester inequalities

$$c_{20} > 0, \quad 4 . c_{20}c_{02} - c_{11}^2 > 0, \quad c_{20}^* > 0, \quad 4 . c_{20}^* c_{02}^* - c_{11}^{*2} > 0$$

Now let us prove the following theorem.

T h e o r e m 3.2. If form (2.11) can assume positive values on even just one half-line of the broken straight lines defined by Eq. (3.1), then system (2.8) cannot be stabilized by means of control (1.7).

P r o o f. Without limiting generality we can assume that this half-line is the positive axis ξ. In this case the following relations must be fulfilled for $\eta = 0$ and $\xi > 0$ for coefficients (2.9)

$$l_{m000}^{(m)} + l_{m010}^{(m)} > 0, \qquad h_{m000}^{(m)} + h_{m010}^{(m)} = 0 \tag{3.6}$$

The Chetaev function corresponding to system (2.8) can be taken in the form

$$2V = \xi^{2k} - \eta^2 \qquad (k > 0 \text{ and is an integer}) \tag{3.7}$$

By virtue of Eq. (2.8) the derivative dV/dt is then given by

$$dV / dt = k\xi^{2k-1} X_m(\xi, \eta) - \eta Y_m(\xi, \eta) + \cdots \tag{3.8}$$

Let us consider the domain $(D): \xi > 0$ and $|\eta| \leq \xi^k$. The function V in the domain

(D) is $V \geq 0$. The sign of the derivative dV/dt is given by Expression

$$k\,(l_{m000}^{(m)} + l_{m010}^{(m)})\,\xi^{2k} - (h_{m-1\,100}^{(m)} + h_{m-1\,110}^{(m)})\,\eta^2$$

In the domain (D) under condition (3.6) even a sufficiently large k is always positive for all coefficients α_{skpq}^{j} (1.7). Thus, the constructed V (3.7) satisfies all the conditions of Chetaev's theorem on instability [3 to 5].

4. Let us consider the case where the form $Q_{m+1}(\xi, \eta)$ is of sign definite. In accordance with the sign definite requirement for functions from the class (2.9), we take (2.11) in the form

$$Q_{m+1}(\xi, \eta) = \sum \gamma_{ij}^{(m+1)} (\xi\xi_*)^i (\eta\eta_*)^j \qquad (i + j = m + 1) \tag{4.1}$$

The coefficients $\gamma_{ij}^{(m+1)}$ in (4.1) which are to be determined will be chosen in such a way as to guarantee fulfillment of the conditions

$$\gamma_{ij}^{(m+1)} > 0 \quad \text{or} \quad \gamma_{ij}^{(m+1)} < 0 \tag{4.2}$$

The homogeneous functions X_i, Y_i, P_{m+1}, Q_{m+1} are here more conveniently written as

$$X_i\,(\xi, \eta, |\xi|, |\eta|),\ Y_i\,(\xi, \eta, |\xi|, |\eta|), \ldots \tag{4.3}$$

By means of the substitution $\xi = r\cos\theta$, $\eta = r\sin\theta$ $(r \geqslant 0)$ we transform system (2.8) for (2.10) into

$$dr\,/\,dt = r^m P_{m+1}(\cos\theta,\ \sin\theta) + r^{m+1}P_{m+2}(\cos\theta,\ \sin\theta) + \ldots \tag{4.4}$$

$$d\theta\,/\,dt = r^{m-1}Q_{m+1}(\cos\theta, \sin\theta) + r^m Q_{m+2}(\cos\theta,\ \sin\theta) + \ldots \tag{4.5}$$

Here $P_{m+i}(\theta)$, $Q_{m+i}(\theta)$ $(i = 1, 2, \ldots)$ are functions of $\cos\theta$, $\sin\theta$ of the form (4.3). Their coefficients depend on α_{skpq}^{j} as in (1.7) and on $\gamma_{ij}^{(m+1)}$ as in (4.1).

Let us first consider the case where

$$g = \frac{1}{2\pi}\int_0^{2\pi} \frac{P_{m+1}(\cos\theta, \sin\theta)}{Q_{m+1}(\cos\theta, \sin\theta)}\, d\theta \neq 0 \tag{4.6}$$

On the basis of one of the properties of periodic functions we have

$$\int_0^{\theta} \frac{P_{m+1}(\cos\theta, \sin\theta)}{Q_{m+1}(\cos\theta, \sin\theta)}\, d\theta = g\theta + f(\theta) \tag{4.7}$$

where $f(\theta)$ is a continuous periodic function by virtue of the sign definite property of the function $Q_{m+1}(\theta)$. We introduce the new variable

$$z = re^{-f(\theta)} \tag{4.8}$$

As our Liapunov function in this case we can take

$$V = z \qquad (z \geqslant 0)$$

By virtue of Eqs. (4.4), (4.5) and (4.8) the derivative dV/dt can be written as

$$dV/dt = gQ_{m+1}(\cos\theta, \sin\theta)\, z^m e^{(m-1)f(\theta)} + z^{m+1}P_{m+1}^{*}(\theta) + \ldots$$

where $P_{m+1}^{*}(\theta)$ is a function of the form (4.3) in $\cos\theta$, $\sin\theta$. We can therefore draw the following conclusion [3 to 5].

Theorem 4.1. Stabilization of system (2.8) is assured by control (1.7) if the coefficients α_{skpq}^{j} and $\gamma_{ij}^{(m+1)}$ can be chosen (provided (2.10), (4.2) are fulfilled) in such a way that the inequality $gQ_{m+1}(\cos\theta, \sin\theta) < 0$ is satisfied. If the inequality $gQ_{m+1}(\cos\theta, \sin\theta) > 0$ is fulfilled for any choice of the coefficients α_{skpq}^{j}, $\gamma_{ij}^{(m+1)}$ then stabilization by means of control (1.7) is impossible.

Now let us consider the case where the constant $g = 0$. Making the substitution $\rho = re^{-f(\theta)}$, we convert system (4.4),(4.5) to the new variable. We then eliminate dt from the resulting equations to obtain

$$\frac{d\rho}{d\theta} = \rho^2 H_2(\theta) + \rho^3 H_3(\theta) + \ldots \qquad (4.9)$$

where $H_k(\theta)$ is a function of $\cos\theta$, $\sin\theta$ of the form (4.3). We shall attempt to find the solution of Eq. (4.9) in series form

$$\rho = c + c^2 u_2(\theta) + c^3 u_3(\theta) + \ldots \qquad (\rho(0, c) = c)$$

Here c is an arbitrary constant and $u_i(\theta)$ is a periodic function of θ defined by Eqs.

$$du_2 / d\theta = H_2(\theta) = R_2(\theta), \qquad du_3 / d\theta = H_3(\theta) + 2u_2(\theta) \quad H_2(\theta) = R_3(\theta), \ldots$$

If $u_k(\theta)$ $(k \leq N-1)$ is the first nonperiodic function in the sequence $u_2, u_3, \ldots$, then it is of the form

$$u_k(\theta) = g^*\theta + G(\theta) \qquad \left(g^* = \frac{1}{2\pi}\int_0^{2\pi} R_k(\theta)\, d\theta \neq 0\right)$$

where $G(\theta)$ is a periodic function.

This immediately implies [1 and 3 to 5] the validity of the following theorem.

Theorem 4.2. If (provided conditions (2.10),(4.2) are fulfilled) the coefficients α^j_{skpq}, $\gamma_{ij}^{(m+1)}$ can be chosen in such a way as to guarantee fulfillment of the inequality $g^* Q_{m+1}(\theta) < 0$, then the stabilization of system (2.8) by control (1.7) is assured. If the inequality $g^* Q_{m+1}(\theta) > 0$ is fulfilled for any chosen coefficients α^j_{skpq}, $\gamma_{ij}^{(m+1)}$, then system (2.8) cannot be stabilized by means of control (1.7).

The author is grateful to V. V. Rumiantsev for his valuable suggestions.

BIBLIOGRAPHY

1. Liapunov, A. M., The General Problem of Stability of Motion. Gostkhizdat, Moscow-Leningrad, 1950.
2. Chetaev, N. G., The Stability of Motion, Gostekhizdat, Moscow, 1955.
3. Malkin, I. G., The Theory of Motion Stability, 2nd ed., "Nauka", Moscow, 1966.
4. Malkin, I. G., Some problems in the theory of motion stability in the Liapunov sense. Sb. Nauchn, Tr. Kasansk. Aviats. Inst., No. 7, 1937.
5. Kamenkov, G. V., On the stability of motion. Sb. Tr. Kazansk. Aviats. Inst. No. 9, 1939.
6. Pliss, V. A., The reduction principle in the theory of motion stability. Dokl. Akad. Nauk SSSR, Vol. 154, No. 5, 1964.
7. Kalman, R. E., On the general theory of control systems. Proceedings of the First Congress of the International Automatic Control Federation, Vol. 2, Isd. Akad. Nauk SSSR, 1961.
8. Gal'perin, E. A. and Krasovskii, N. N., The stabilization of stationary motions in nonlinear control systems. PMM Vol. 27, No. 6, 1963.
9. Gal'perin, E. A., On the stabilization of steady-state motions of a nonlinear control system in the critical case of a pair of pure imaginary roots. PMM Vol. 29, No. 6, 1965.
10. Stoianov, N. V., On the stabilization of the steady-state motion of a nonlinear control system in the critical case of a double zero root. PMM Vol. 31, No. 3, 1967.

Translated by A. Y.